

SYMMETRY IN MAXIMAL $(s - 1, s + 1)$ CORES

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ABSTRACT. We explain a “curious symmetry” for maximal $(s - 1, s + 1)$ -core partitions first observed by T. Amdeberhan and E. Leven. Specifically, using the s -abacus, we show such partitions have empty s -core and that their s -quotient is comprised of 2-cores. This imposes strong conditions on the partition structure, and implies both the Amdeberhan-Leven result and additional symmetry. We also find a more general family of partitions that exhibits these symmetries.

1. INTRODUCTION

The study of simultaneous core partitions, which began only fifteen years ago, has seen a recent spike of interest. Much of the attention has focused around either a conjecture of Armstrong on the average size of an (s, t) -core or generalizing known results on the $(s, s + 1)$ (Catalan) case. Results in a recent paper of Amdeberhan and Leven deviate from this slightly to examine $(s - 1, s + 1)$ -cores in the case where s is even and greater than 2; they note a symmetry in the set of first column hook numbers of $\kappa_{s \pm 1}$, the $(s - 1, s + 1)$ -core of maximal size. [Theorem 2.2 in this paper states their result.]

Hidden by their proof (which involves the integral and fractional parts of a real number) is a connection with the s -core and s -quotient structure viewed on the s -abacus. From this vantage point, the Amdeberhan-Leven theorem is a result on the symmetry of runners (columns) of the s -abacus of maximal $(s - 1, s + 1)$ -cores.

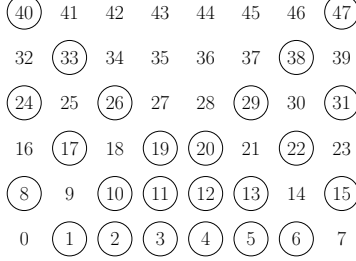
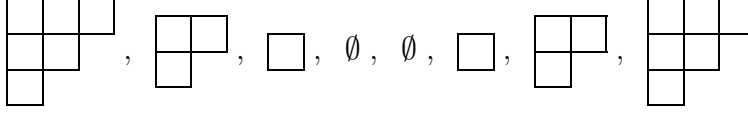
Given a partition λ , let λ^0 be the s -core of λ and $(\lambda_{(0)}, \lambda_{(1)}, \dots, \lambda_{(s-1)})$ be the s -quotient of λ . Let $\kappa_{s \pm 1}$ be the unique maximal simultaneous $(s - 1, s + 1)$ -core partition and $\tau_\ell = (\ell, \ell - 1, \ell - 2, \dots, 1)$ be the ℓ -th 2-core partition. We state our main theorem.

Theorem 1.1. *Let $s = 2k > 2$. Then $\kappa_{s-1, s+1}$ has the following s -core and s -quotient structure:*

$$(1) \ (\kappa_{s-1, s+1})^0 = \emptyset.$$

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FIGURE 1. The 8-abacus of $\kappa_{7,9}$ FIGURE 2. 8-quotient of $\kappa_{7,9}$ 

$$(2) \quad \kappa_{s\pm 1(i)} = \kappa_{s\pm 1(s-i-1)} = \tau_{k-i-1} \text{ where } 0 < i < k-1.$$

In Section 3.1 we describe the s -abacus of $\kappa_{s\pm 1}$, which we use to prove Theorem 1.1. We provide an alternate proof of the Amdeberhan-Leven result in Section 3.2. In Section 4.1 we demonstrate an additional symmetry in the rows of the s -abacus of $\kappa_{s-1,s+1}$. We formalize both the runner and row symmetries exhibited by $\kappa_{s\pm 1}$ in Section 4.2, and describe the most general family of partitions which satisfy them.

Example 1.2. The 8-abacus of $\kappa_{7,9}$ and the associated 8-quotient are shown in **Figure 1** and **Figure 2** respectively. [Note: the 8-quotient consists of a sequence of 2-core partitions, arising from the structure of the 8-abacus.]

2. PRELIMINARIES

2.1. Basic definitions. Let $\mathbb{N} = \{0, 1, \dots\}$ and $n \in \mathbb{N}$. A *partition* λ of n is defined as a finite, non increasing sequence of positive integers $(\lambda_1, \lambda_2, \dots)$ that sums to n . Each λ_γ is known as a *component* of λ . Then $\sum_\gamma \lambda_\gamma = n$, and λ is said to have *size* n , denoted $|\lambda| = n$. We also use the notation λ_i^m to indicate that λ_i occurs m -times as a component of λ .

The *Young diagram* $[\lambda]$ is a graphic representation of λ in which rows of boxes corresponding to the integer values in the partition sequence are left-aligned. Then λ^* is the *conjugate partition* of λ obtained by

7	4	2	1
4	1		
2			
1			

FIGURE 3. Young diagram (with hook lengths) of $\kappa_{3,5}$

exchanging rows and columns of the Young diagram of λ . Then λ is *self-conjugate* if $\lambda = \lambda^*$. Using matrix notation, a *hook* $h_{\iota\gamma}$ of $[\lambda]$ with *corner* (ι, γ) is the set of boxes to the right of (ι, γ) in the same row, below (ι, γ) in the same column, and (ι, γ) itself. Given $h_{\iota\gamma}$, its *length* $|h_{\iota\gamma}|$ is the number of boxes in the hook. The set $\{h_{1\gamma}\}$ are the *first-column hooks* of λ .

One can *remove* a hook h of λ by deleting boxes in $[\lambda]$ which comprise h and migrating any remaining detached boxes up-and-to-the-left. In this way a new partition λ' of size $n - |h_{\iota\gamma}|$ is obtained. An *s-hook* is a hook of length s . An *s-core partition* λ is one in which no hook of length s appears in the Young diagram.

2.2. Simultaneous (s, t) -core partitions. Let r, s, t be positive integers. A *simultaneous (s, t) -core partition* is one in which no hook of length s or t appears. In 1999, J. Anderson [5] proved when $(s, t) = 1$, there are exactly $\binom{s+t}{t}/(s+t)$ simultaneous (s, t) -cores. Subsequent work by B. Kane [10], J. Olsson and D. Stanton [13], J. Vandehey [16] confirmed the existence of a unique *maximal (s, t) -core* of size $\frac{(s^2-1)(t^2-1)}{24}$ which contains all other (s, t) -cores. This maximal simultaneous (s, t) -core partition is denoted by $\kappa_{s,t}$. [A. Tripathi [15] and M. Fayers [8] obtained some of the results above using different methods.] When it is convenient we will denote $\kappa_{s-1, s+1}$ by $\kappa_{s\pm 1}$.

Theorem 2.1. [Olsson-Stanton, Theorem 4.1, [13]] Suppose $(s, t) = 1$. There is a unique maximal simultaneous (s, t) -core $\kappa_{s,t}$ of size $\frac{(s^2-1)(t^2-1)}{24}$. In particular, $\kappa_{s,t}$ is self-conjugate.

A recent paper of D. Armstrong, C. Hanusa and B. Jones [6] includes a conjecture (the Armstrong conjecture) that the average size of a (s, t) -core is $\frac{(s+t+1)(s-1)(t-1)}{24}$. R. Stanley and F. Zenello [14] subsequently resolved the Catalan ($t = s + 1$) case of the Armstrong conjecture; they employ a bijection between lower ideals in the poset $P_{s,t}$ and simultaneous (s, t) -cores. [Here $P_{s,t}$ is the partially ordered set whose

						47	
33					38		40
				29		31	
17	18	19	20		22		24
	10	11	12	13		15	
1	2	3	4	5	6		8

FIGURE 4. Amdeberhan-Leven rectangle R for $P_{7,9}$

elements are all positive integers not contained in the numerical semi-group generated by s, t . The partial order requires $z_1 \in P_{s,t}$ to cover $z_2 \in P_{s,t}$ if $z_1 - z_2$ is either s or t .] Under this map a lower ideal I of $P_{s,t}$ corresponds to an (s, t) -core partition whose first-column hook lengths are exactly the values in I . Then $P_{s,t}$ corresponds to $\kappa_{s,t}$.

These two papers have led to renewed interest in simultaneous core partitions. The Armstrong conjecture has been verified for self-conjugate partitions by W. Chen, H. Huang, and L. Wang [7] and for $(s, ms + 1)$ by A. Aggarwal [1]. T. Amdeberhan and E. Leven [4] extended Stanley and Zanello's bijection to lower poset ideals and simultaneous (s_1, s_2, \dots, s_k) -cores. Several conjectures of T. Amdeberhan [5] on the maximal and average size simultaneous $(s, s + 1, s + 2)$ -cores have been proved first by J. Yang, M. Zhong and R. Zhou [18] and later by H. Xiong [17]. A. Aggarwal has also proved a partial converse to a theorem of Vandehey on the containment of simultaneous (r, s, t) -cores [2].

2.3. A “curious symmetry”. Amdeberhan and Leven also examine $P_{r,r+2}$ for r odd. They first construct a $(r - 1) \times (r + 1)$ rectangle R as follows: the bottom-left corner is labelled by 1, the numbers increase from left-to-right and bottom-to-top, and the largest position, in the upper-right corner, is labeled by $(r - 1)(r + 1)$. If $x \in P_{r,r+2}$ then x is entered into this rectangle, otherwise the position is left blank. Using a *runner-row index*, counting runners (or columns) a from left-to-right in the x -coordinate ($1 \leq a \leq r + 1$), and rows b from bottom-to-top in the y -coordinate ($1 \leq b \leq r - 1$), they prove the following result, which they call a “curious symmetry.”

Theorem 2.2. [Amdeberhan-Leven, Theorem 2.2, [4]] *For $r \geq 3$ the (a, b) entry of R is an element of $P_{r,r+2}$ if and only if $\{a, r - 1 - b\}$ is not. Equivalently, for $1 \leq a \leq r + 1$ and $1 \leq b \leq r - 1$, $(r + 1)(b - 1) + a \in P_{r,r+2}$ if and only if $(r + 1)(r - 1 - b) + a \notin P_{r,r+2}$.*

[There is a precedent for the case Amdeberhan-Leven consider. For $r = 2k+1 > 1$, the maximal simultaneous $(r, r+2)$ -core is self-conjugate by Theorem 2.1. The author and C. Hanusa showed in [10] that it is more natural to think about simultaneous $(r, r+2)$ -core partitions than simultaneous $(s, s+1)$ -core partitions, which behave better in the non-self-conjugate case.] For the remainder of this paper we will let $s = r+1$, and will consider maximal $(s-1, s+1)$ -core, where s is even and greater than 2. We now review the s -abacus, s -core, and s -quotient constructions.

2.4. bead-sets. A *bead-set* X corresponding to a partition λ is generalization of the set of first column hooks in the following sense: $X = \{0, 1, \dots, k, |h_{11}|+k, |h_{12}|+k, |h_{13}|+k, \dots\}$ for some non-negative integer k . It can also be seen as a finite set of non-negative integers, represented by *beads* at integral points of the x -axis, i.e. a bead at position x for each x in X and *spacers* at positions not in X . A *minimal* bead-set X is one where the first space is counted as 0. Then $|X|$ is the number of beads that occur after the zero position, where ever that may fall. We say $X = \{0, 1, \dots, k, |h_{11}|+k, |h_{12}|+k, |h_{13}|+k, \dots\}$ is *normalized with respect to s* if k is the minimal integer such that $|X| \equiv 0 \pmod{s}$.

2.5. 2-cores and staircase partitions. [The results in this section are stated without proof; for more details see Section 2 in [12].] The set of hooks $\{h_{i\gamma}\}$ of λ correspond bijectively to pairs (x, y) where $x \in X$, $y \notin X$ and $x > y$; that is, a bead in the bead-set X of λ and a spacer to the left of it. Hooks of length s are those such that $x - y = s$.

Each first-column hook length, or bead x_i in the minimal bead-set X , also corresponds to a row, or component λ_i of λ . The following result allows us to recover the size of the components from X .

Lemma 2.3. The size of the component λ_i corresponding to the bead $x_i \in X$ is the number of spacers to the left of the bead; that is, $\lambda_i = |y \notin X : y < x_i|$.

Let $\tau_k = (k, k-1, \dots, 1)$ be the k -th *staircase partition*. Then $|\tau_k| = t_k$ where $t_k = \binom{k+1}{2}$ (the k -th triangular number). The following lemmas are well-known.

Lemma 2.4. The 2-core partitions are exactly the staircase partitions.

Lemma 2.5. The minimal X for the 2-core τ_k is $\{1, 3, 5, \dots, 2k-3, 2k-1\}$. In other words, the 2-core partitions are sequence of alternating spacers-and-beads of length $2k-1$.

2.6. The s -abacus. Given a fixed integer s , we can arrange the non-negative integers in an array of columns and consider the columns as runners.

$$\begin{array}{cccc}
 ms & & & (m+1)s-1 \\
 \vdots & & \ddots & \\
 s & s+1 & & 2s-1 \\
 0 & 1 & \cdots & s-1
 \end{array}$$

The column containing i for $0 \leq i \leq s-1$ will be called *runner i* . The positions $0, 1, 2, \dots$ on the i th runner corresponding to $i, i+s, i+2s, \dots$ will be called *i -positions*. Placing a bead at position x_j for each $x_j \in X$ gives the *s -abacus diagram* of X . A *normalized* abacus will be one whose bead-set X is normalized, a *minimal* abacus is one in which X is minimal (or, the first spacer is counted as the zero position).

2.7. The s -core and s -quotient. By removing a sequence of s -hooks from λ until no s -hooks remain, one obtains its *s -core* λ^0 . The *s -abacus* of λ^0 can be found from the *s -abacus* of λ by pushing beads in each runner down as low as they can go (Theorem 2.7.16,[9]: we have changed the orientation). This implies λ^0 is unique since it is independent of the way the s -hooks are removed. For $0 \leq i \leq p-1$ let $X_i = \{j : i + js \in X\}$ and let $\lambda_{(i)}$ be the partition represented by the bead-set X_i . The *s -quotient* of λ is the sequence $(\lambda_{(0)}, \dots, \lambda_{(s-1)})$ obtained from X . The next lemma is Proposition 3.5 in [12].

Lemma 2.6. Let λ be a partition with s -core λ^0 and s -quotient $(\lambda_{(i)})$, $0 \leq i \leq s-1$. Then

- (1) Every 1-hook in $\lambda_{(i)}$ corresponds to a s -hook in λ for $0 \leq i \leq s-1$.
- (2) $n = |\lambda^0| + \sum_i |\lambda_{(i)}|$.

Lemma 2.6 implies that there exists a bijection between a partition λ and its s -core and s -quotient, such that each node in some λ_i corresponds to an s -hook in λ . The situation is strengthened when λ is self-conjugate.

Lemma 2.7. Suppose $|X| \equiv 0 \pmod{s}$. Let λ^* be the conjugate of λ , $(\lambda^*)^0$ its s -core and let (λ_i^*) be the s -quotient of λ^* , $0 \leq i \leq s-1$. Then

- (1) $(\lambda^*)^0 = (\lambda^0)^*$
- (2) $(\lambda_{(i)}^*)^* = \lambda_{(s-1-i)}$.

In particular, $\lambda = \lambda^*$ if and only if $\lambda^0 = (\lambda^0)^*$ and $(\lambda_{(i)})^* = (\lambda^*)_{(i)}$.

2.8. The axis of symmetry. The following results and their proofs can be found in Section 4, [11].

Proposition 2.8. Suppose λ is a partition of n and let X be a bead-set for λ . Then there exists a half-integer $\theta(\lambda)$ such that the number of beads to the right of $\theta(\lambda)$ equals the number of spaces to its left. Conversely, given a bead-spacer sequence and a half-integer $\theta(\lambda)$ such that the number of beads to the right equals the number of spaces to the left, one can recover the unique partition λ .

Lemma 2.9. Let X be a minimal bead-set for λ . If $x' \in X$ is the entry with maximum value, $\theta(\lambda) = \frac{x'}{2}$.

We call $\theta(\lambda)$ the *axis* of λ . If λ is self-conjugate we say X has a *axis of symmetry*.

Corollary 2.10. Let X be a bead-set for λ . Then λ is a self-conjugate partition if and only if there exists a half-integer $\theta(\lambda)$ such that beads and spaces in X to the right of $\theta(\lambda)$ are reflected respectively to spaces and beads in X to its left.

When $\lambda^0 = \emptyset$, each λ_i has an axis of symmetry $\theta(\lambda_i)$ induced by X .

Lemma 2.11. Suppose X is normalized. Then $|X| = ms$, $\lambda^0 = \emptyset$, and each runner has exactly m beads if and only if $\theta(\lambda_{(i)}) = \theta(\lambda_{(i')}) = m - \frac{1}{2}$ for all $0 \leq i, i' \leq s-1$.

Example 2.12. The maximum $(5, 7)$ -core $\kappa_{5,7}$ has empty 8-core. In the normalized (minimal) 8-abacus in **Figure 1**, each $\lambda_{(i)}$ has axis $\theta(\lambda_{(i)}) = \frac{5}{2}$.

3. THE s -QUOTIENT OF $\kappa_{s\pm 1}$

3.1. The s -abacus of $\kappa_{s\pm 1}$. We begin with a classical result of Sylvester.

Lemma 3.1. The largest integer in $P_{s,t}$ is $st - s - t$.

The Amdeberhan-Leven rectangle R is constructed to begin at 0; the $(r+1)$ -abacus of $\kappa_{r,r+2}$ starts at 0. However $0 \notin P_{r,r+2}$ and by Lemma 3.1 neither is $(r+1)(r-1)$. Hence R and the minimal $(r+1)$ -abacus of $\kappa_{r,r+2}$ include the same values.

Recall $s = r+1$. We now interpret the Amdeberhan-Leven result in terms of the s -abacus $\kappa_{s-1,s+1}$. We use a runner-row index. We start with a definition.

Definition 3.2. Let $s = 2k > 2$. Then $\alpha(s)$ is an s -abacus with s runners, indexed from left-to-right by $0 \leq i \leq s-1$ and $s-2$ rows,

indexed from bottom-to-top by $0 \leq i \leq s-3$, which is constructed as follows: For each $i \in [0, k-2]$, the runners i and $2k-i-1$ are composed firstly of beads in rows j where $0 \leq j \leq i$. Then rows $j > i$ consist of alternating spacers-and-beads, until the total number of beads in each runner is $(k-1)$. Spacers fill the remainder of the rows.

Example 3.3. $\alpha(8)$ has three beads in each runner. Runners $i=3$ and 4 consist of three beads below three spacers; $i=2$ and 5 have two beads followed by a spacer-and-bead, then two spacers; $i=1$ and 6 have one bead followed by spacer-bead-spacer-bead-spacer; and runners $i=0$ and 7 have an alternating sequence of spacers-and-beads. [See **Figure 1**.]

Lemma 3.4. The s -abacus $\alpha(s)$ is normalized with respect to s .

Proof. The total number of beads in $\alpha(s)$ is $2k(k-1) = \frac{s^2-2s}{2}$, a multiple of s . \square

Lemma 3.5. Fix $1 < j < 2k-3$ and $0 \leq i < k-1$.

- (1) There is a bead in row j of runner 0 if and only if there is a bead in row $j-1$ of runner 1.
- (2) There is a bead in row j of runner $2k-1$ if and only if there is a bead in row $j-1$ of runner $2k-2$.
- (3) There is a spacer in row j of runner 0 if and only if there is a spacer in row $j+1$ of runner 1.
- (4) There is a spacer in row j of runner $2k-1$ if and only if there is a spacer in row j of runner $2k-2$.

Proof. By Definition 3.2, runner $i=0$ begins in row $j=0$ with a spacer, and continues upwards with alternating beads-and-spacers. Runner $i=1$ begins with a bead in row 1, and continues upwards, alternating spacers-and-beads. Since both columns have $2k-2$ rows, (1) and (3) follow. For (2) and (4), a similar argument holds. \square

Lemma 3.6. The s -abacus $\alpha(s+2)$ can be obtained from the s -abacus $\alpha(s)$ using the following procedure:

- (1) Append a new row of $2k$ beads below $\alpha(s)$.
- (2) Append a new row of $2k$ spacers above $\alpha(s)$.
- (3) Append a new runner of length $2k-2$ consisting of alternating beads-and-spacers to the *left* (and an identical column to the *right*) of $\alpha(s)$.
- (4) Append a single spacer to the bottom, and a single bead at the top of, both new runners in step (3). [The total number of beads in all runners, both the two new runners, as well as the $s=2k$ previous runners, will now be k .]

- (5) Renumber the runners with i' so $0 \leq i' \leq 2k+1$ and the rows with j' so that $0 \leq j' \leq 2k-1$. Renumber the abacus positions, with 0 in the bottom left-most corner, increasing from left-to-right and bottom-to-top, with final position $(2k+1)(2k-1)$ in the upper-right-hand corner.

Proof. It is enough to see that the outcome satisfies Definition 3.2 for $\alpha(s+2)$. \square

Example 3.7. To see how Lemma 3.6 is used to obtain $\alpha(10)$ from $\alpha(8)$, see **Appendix A, Figure 9** and **Figure 8**.

Recall λ^0 is the s -core partition of λ , $(\lambda_{(i)})$ is the s -quotient (where $0 \leq i \leq s-1$), and that τ_ℓ the ℓ -th 2-core partition. For the following two lemmas we abuse notation and let $\alpha(s)$ refer not only to the s -abacus but also to its corresponding partition.

Lemma 3.8. Suppose $s = 2k > 2$. Then

- (1) $\alpha(s)^0 = \emptyset$
- (2) $\alpha(s)_{(i)} = \alpha(s)_{(s-i-1)} = \tau_{k-i+1}$.

Proof. We proof each condition separately.

- (1) Since each runner $\alpha(s)_i$ has $k-1$ beads and $(k-1)$ spacers, the removal of all s -hooks will result in an s -abacus with each runners having $k-1$ beads beneath $k-1$ spacers. This corresponds to the empty partition.
- (2) We use induction on k . For $k = 2$ it is true. Assume it is for k . We obtain the $\alpha(s+2)$ from $\alpha(s)$ by Lemma 3.2. By construction, for $1 \leq i' \leq 2k$ we have $|\alpha(s)_{(i'-1)}| = |\alpha(s+2)_{(i')}|$; hence, by the inductive hypothesis and since $i+1 = i'$, $|\alpha(s+2)_{(i')}| = \tau_{(k+1)-i'-1}$. It only remains to check $i' = 0, 2k+1$. The proof is finished using (3) and (4) of Lemma 3.6 and Lemma 2.5. \square

Example 3.9. $\alpha(8)$ has 8-quotient $(\lambda_0, \dots, \lambda_{s-1})$

$$(3, 2, 1), (2, 1), (1), \emptyset, \emptyset, (1), (2, 1), (3, 2, 1))$$

[See **Appendix A, Figure 8** and **Appendix B, Figure 12**]

Lemma 3.10. Let $s = 2k > 2$. Then $\alpha(s)$ is the minimal s -abacus for $\kappa_{s-1, s+1}$.

Proof. By construction, $\alpha(s)$ is minimal, since the first spacer labels zero. We must show:

- (1) $|\alpha(s)| = \frac{((2k-1)^2-1)((2k+1)^2-1)}{24}$,

(2) $\alpha(s)$ contains no $(2k - 1)$ -hooks or $(2k + 1)$ -hooks.

Then by the uniqueness implied by Theorem 2.1, $\alpha(s) = \kappa_{s \pm 1}$. We use the structure of $\alpha(s)$ and induction on k .

By Theorem 2.6 each 1-hook in the s -quotient corresponds to a s -hook in λ . Hence, to prove (1), since $\alpha(s)^0 = \emptyset$, it is enough to calculate $\sum_i |\lambda_{(i)}|$ and multiply by $s = 2k$. This equals $2k \cdot 2 \sum_{i=1}^{k-1} t_i = (4k) \frac{(k-1)(k)(k+1)}{6}$. In particular $4k \frac{(k)(k^2-1)}{6} = \frac{16k^4-16k^2}{6} = \frac{(4k^2-4k)(4k^2+4k)}{24}$, which, after completing-the-square, equals to $\frac{((2k-1)^2-1)((2k+1)^2-1)}{24}$. We are done.

To prove (2), we use induction on $k > 2$. For the basic case, $s=4$, it holds: $\alpha(4)$ has no 3-hooks or 5-hooks. [See **Appendix A, Figure 6.**]

By the inductive hypothesis we know the $2k$ -abacus of $\kappa_{2k \pm 1}$ contains no $(2k - 1)$ -hooks or $(2k + 1)$ -hooks. More specifically, no bead in $\alpha(s)$ has a spacer either $2k + 1$ or $2k - 1$ positions below it. Apply Lemma 3.6 to obtain $\alpha(s + 2)$; this adds two additional positions between the beads and spacers arising from $\alpha(s)$. Hence there are no $(2k + 1)$ -hooks or $(2k + 3)$ -hooks arising from bead-spacer pairs (x, y) where both x and y are in runners $1 < i' < 2k - 2$. It remains to examine the beads and spacers introduced by runners $i' = 0, 2k + 1$.

If a bead in row j' of runner $i' = 0$ were to add a new $(2k + 3)$ -hook, a spacer would have to appear in row $j' - 2$ of the runner $i' = 2k + 1$. By construction, such positions are occupied by beads, since runners 0 and $2k + 1$ are identical. If a bead in row j' of $i' = 0$ were to add a new $(2k + 1)$ -hook, a spacer would have to appear in row $j' - 1$ of runner $i' = 1$. But by the Lemma 3.5(1), this position is always occupied by a bead.

If a bead in row j' on runner $i' = 2k + 1$ were to add a new $(2k + 3)$ -hook, a spacer would appear in row $j' - 1$ of runner $i' = 2k$. But by Lemma 3.5(2) this position is always occupied by a bead. If a bead in row j' of runner $i' = 2k + 1$ were to add a new $(2k + 1)$ -hook, a spacer would have to appear in the same row in the runner $i' = 0$. By construction, the two runners are identical, so a bead in one implies a bead in the other.

If a spacer in row j' of runner $i' = 0$ were to add a new $(2k + 3)$ -hook, a bead would have to appear in row $j' + 1$ of runner $i' = 1$. But by Lemma 3.5(3), this position is always occupied by a spacer. If a spacer in row j' of $i' = 0$ were to add $(2k + 1)$ -hook, a bead would have to appear in the same row of runner $i' = 2k + 1$. By construction, the two runners are identical, so a spacer in one implies a spacer in the other.

If a spacer in row j' of runner $i' = 2k+1$ were to add a new $(2k+3)$ -hook, a bead would have to appear in row $j' + 2$ in runner $i' = 0$; by construction, since both runners are identical alternating sequences of spacer-and-beads, such positions are occupied by spacers. If a spacer in row j' of runner $i' = 2k+1$ were to add a new $(2k+1)$ -hook, a bead would have to appear in row $j' + 1$ of runner $i' = 2k$. But by Lemma 3.5(4) this position is occupied by a spacer. \square

3.2. An alternative proof of Amdeberhan-Leven. Using the results of this section we offer an alternative proof to Theorem 2.2.

Proof of Theorem 2.2. By Lemma 3.10, the s -core of $\kappa_{s\pm 1} = \emptyset$, and X is normalized. Again by Lemma 3.10, each λ_i is self-conjugate, so each runner obeys Lemma 2.10. By Lemma 2.11, all $(\kappa_{s-1, s+1})_i$ have the same axis of symmetry, which is at i -position $\frac{s-3}{2}$. Our runner-row index is $0 \leq j \leq s-3$ with $s = r-1$, which finishes the proof. \square

4. GENERALIZATIONS

4.1. Additional symmetry. Using Theorem 1.1 we can strengthen Amdeberhan-Leven to include additional symmetry.

Theorem 4.1. *Let $s = 2k > 2$ and let $\alpha(s)$ be the s -abacus of $\kappa_{s\pm 1}$. Then the following are equivalent:*

- (1) $(i, j) \in \alpha(s)$
- (2) $(i, s-3-j) \notin \alpha(s)$
- (3) $(s-1-i, j) \in \alpha(s)$.

Proof. By Theorem 2.2 is sufficient to prove (1) \iff (3). This follows from Lemma 3.10 and Lemma 3.8. \square

4.2. Horizontal anti-symmetry and vertical symmetry. The symmetries exhibited by the s -abacus of $\kappa_{s\pm 1}$ can be formalized and generalized to a larger family of partitions. For the remainder of this section we assume that the bead-set X of λ is normalized with respect to s . Suppose that the s -abacus of λ has maximum value $i + (q-1)s$. In particular, the s -abacus of λ has s columns and q rows.

Definition 4.2. We say the s -abacus of λ exhibits *horizontal anti-symmetry* if there is a bead in the (i, j) th-position if and only if there is a spacer in the $(i, q-j-1)$ position.

Definition 4.3. We say the s -abacus of λ exhibits *vertical symmetry* if there is a bead in the (i, j) th-position if and only if there is a bead in the $(s-i-1, j)$ th-position.

Lemma 4.4. The s -abacus of λ exhibits horizontal anti-symmetry if and only if q is even, $\lambda_{(i)} = \lambda_{(i)}^*$, and each runner has q beads.

Proof. Suppose the s -abacus of λ exhibits horizontal anti-symmetry. Clearly q must be even, otherwise there would exist a bead or spacer in a row that would not have a spacer or bead to pair with. Let $q = 2m$. Horizontal symmetry also implies each runner i must have the same axis of symmetry, $\theta(\lambda_i) = \frac{q-1}{2} = m - \frac{1}{2}$. Lemma 2.10 implies $\lambda = \lambda^*$. By Lemma 2.11, each runner has exactly q beads. The proof in the other direction is clear. \square

Lemma 4.5. The s -abacus of λ exhibits vertical symmetry if and only if s is even, runner i and runner $s - i - 1$ have the same number of beads, and $\lambda_i = \lambda_{s-i-1}$ for $0 \leq i \leq s - 1$.

Proof. Suppose the s -abacus of λ exhibits horizontal symmetry. Then s must be even, otherwise there would a bead or spacer in a runner that would not have a bead or spacer to pair with. Vertical symmetry also implies that each runner i and $s - i - 1$ must be identical. This means runners i and $s - i - 1$ have the same number of beads and $\lambda_i = \lambda_{s-i-1}$ for each $0 \leq i \leq s - 1$. The proof in the other direction is clear. \square

Theorem 4.6. λ exhibits both horizontal anti-symmetry and vertical symmetry with respect to s if and only if s and q are both even and the following three conditions hold for all $0 \leq i \leq s - 1$

- (1) $\lambda^0 = \emptyset$
- (2) $\lambda_{(i)} = \lambda_{(i)}^*$
- (3) $\lambda_{(i)} = \lambda_{(s-i-1)}$.

Proof. This follows from Lemma 4.4 and Lemma 4.5. \square

Example 4.7. $\lambda = (8, 6^4, 1^2)$ exhibits horizontal anti-symmetry and vertical symmetry with respect to $s = 4$, but is neither a 3-core nor a 5-core. See **Figure 5**.

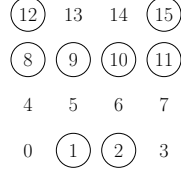
The following corollary is immediate.

Corollary 4.8. Let $s = 2k > 1$. The s -abacus of $\kappa_{s\pm 1}$ exhibits horizontal anti-symmetry and vertical symmetry.

Corollary 4.9. If the s -abacus of λ exhibits both horizontal anti-symmetry and vertical symmetry then λ is self-conjugate.

Proof. By Theorem 4.6, since $\lambda_{(i)} = \lambda_{(s-i-1)}$ and $\lambda_{(i)} = \lambda_{(i)}^*$, we have $\lambda_{(i)} = \lambda_{(s-i-1)}^*$. Since $\lambda^0 = \emptyset$, and by assumption $|X| = 0 \pmod{s}$, we have $\lambda = \lambda^*$ by Lemma 2.7. \square

FIGURE 5. The minimal 4-abacus of $\lambda = (8, 6^4, 1^2)$ (see Example 4.7)



5. FURTHER STUDY

5.1. Simultaneous $(\mathbf{s} - \mathbf{1}, \mathbf{s}, \mathbf{s} + \mathbf{1})$ -cores. The following theorem is a recently-proven conjecture of Amdeberhan [5].

Theorem 5.1. (*Yang-Zhong-Zhou*, [18]; *H. Xiong*, [17]) *The size of the largest $(s - 1, s, s + 1)$ -core is*

- (1) $k \binom{k+1}{3}$ if $s = 2k > 2$
- (2) $(k + 1) \binom{k+1}{3} + \binom{k+2}{3}$ if $s = 2k + 1 > 2$.

Let $\kappa_{s-1,s,s+1}$ is a (not necessarily unique) simultaneous $(s - 1, s, s + 1)$ -core of maximal size. Theorem 5.1 allows us to compare $|\kappa_{s\pm 1}|$ with $|\kappa_{s-1,s,s+1}|$.

Proposition 5.2. Let $s = 2k > 2$. Then $|\kappa_{s\pm 1}| > |\kappa_{(s-1,s,s+1)}|$. In particular, $|\kappa_{s\pm 1}| = 4|\kappa_{(s-1,s,s+1)}|$

Proof. Since s is even, by Theorem 5.1(1) above $|\kappa_{s-1,s,s+1}| = \frac{k^4 - k^2}{6}$. However by Theorem 2.1, $|\kappa_{s\pm 1}| = \frac{((s-1)^2 - 1)((s+1)^2 - 1)}{24}$. This simplifies to $\frac{4(k^4 - k^2)}{6}$. The result follows. \square

Corollary 5.3. $\kappa_{s,s+2}$ is never an s -core.

Corollary 5.3 also follows from Theorem 1.1 which says $\kappa_{s,s+2}$ is comprised completely of s -hooks. Is there interpretation (either in the geometry of the s -abacus or in the manipulation of Young diagrams) of the factor of 4 that appears above? A cursory examination of $\kappa_{(3,5)}$ and $\kappa_{(3,4,5)}$ does not suggest an obvious one.

5.2. Other proofs using the s -abacus. In their proof of Theorem 2.2 Amdeberhan and Leven use the following result (Corollary 2.1 (ii), [4]).

Lemma 5.4. Exactly half of the integers in $\{1, 2, \dots, (s - 1)(t - 1)\}$ belong to $P_{s,t}$.

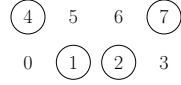
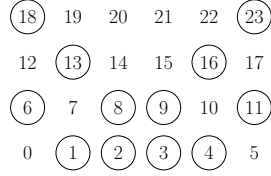
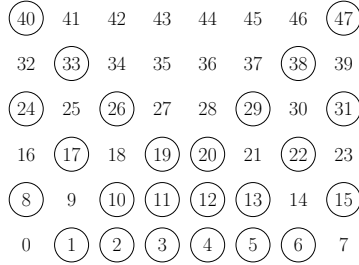
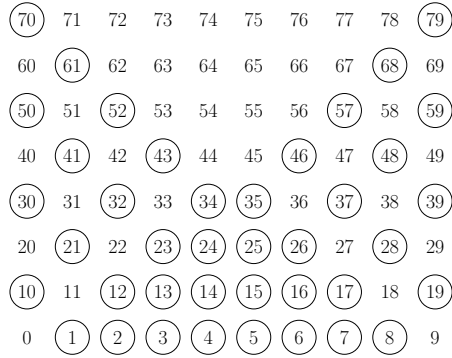
They cite a result of T. Popoviciu on the integral and fractional parts of an integer of which this is a consequence. We provide an alternative proof using only the geometry of the s -abacus.

Proof of Lemma 5.4. Since by Lemma 3.1 neither $(s-1)(t-1)$ nor 0 are in $P_{s,t}$, it is equivalent to prove half of the integers in $\{0, 1, 2, \dots, st - s - t\}$ are in the minimal bead-set of $\kappa_{s,t}$. By Lemma 2.8, the axis is $\theta(\kappa_{s,t}) = \frac{st-s-t}{2}$. This implies the result. \square

Perhaps there are other results on simultaneous core partitions that can be understood using bead-sets and the geometry of the s -abacus.

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APPENDIX A

The s -abaci $\alpha(s)$ of $\kappa_{s\pm 1}$ FIGURE 6. $s = 4$ FIGURE 7. $s = 6$ FIGURE 8. $s = 8$ FIGURE 9. $s = 10$ 

APPENDIX B

The s -quotients of $\kappa_{s\pm 1}$

FIGURE 10. 4-quotient of $\kappa_{3,5}$

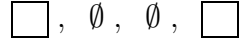


FIGURE 11. 6-quotient of $\kappa_{5,7}$

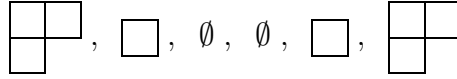


FIGURE 12. 8-quotient of $\kappa_{7,9}$

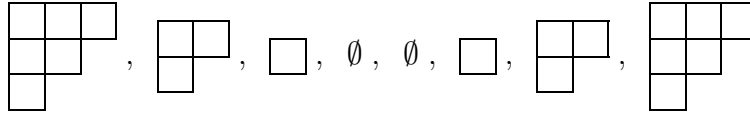
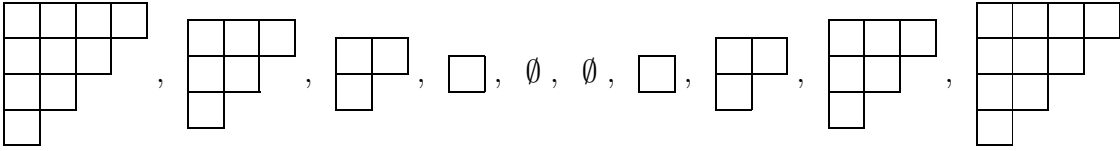


FIGURE 13. 10-quotient of $\kappa_{9,11}$



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